EQUATIONS OF PERTURBED MOTION IN THE KEPLER PROBLEM

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Equations of perturbed motion of a planet were partly known to Newton; the history of the problem and the derivation of these equations are presented in Tisserand's well-known treatise on celestial mechanics [1] and in the work of Krylov [2]. Tisserand, following the general methods of the theory of perturbed motion, computes Lagrange's bracket expressions for the elliptic elements of the orbit; Krylov's derivation is based on geometric constructions. These equations have also been derived in Duboshin's book [3].

The derivation suggested below is based on the direct application of the method of variation of parameters. The equation of the elliptic orbit is written down in vector form.

$$\mathbf{r} = \frac{a(1-e^2)}{1+e\cos\varphi}\,\mathbf{e}_r = r\mathbf{e}_r \qquad (1)$$

where $\mathbf{e}_r$ is the unit vector from the center of attraction to the moving point; a, e are the major semi-axis and the maximum eccentricity of the orbit, $\cos\phi = \mathbf{e}_r\cdot\mathbf{i}_1$, where $\mathbf{i}_1$ is the unit vector in the direction towards the perigee (the major semi-axis of the orbit).

We introduce an orthogonal set of unit vectors $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_3 = \mathbf{e}_r\times\mathbf{e}$; the unit vector $\mathbf{e}_\phi$ is in the orbit plane in the direction of increase of angle ϕ, perpendicularly to $\mathbf{e}_r$, the vector $\mathbf{e}_3$ defines the orbit plane in an unperturbed motion.

In an unperturbed motion this set has an angular velocity $\dot\phi\mathbf{e}_3$, so that

$$\dot{\mathbf{e}}_r = \dot\varphi\mathbf{e}_\varphi, \qquad \dot{\mathbf{e}}_\varphi = -\dot\varphi\mathbf{e}_r, \qquad \dot{\mathbf{e}}_3 = 0 \qquad (2)$$

and according to the law of areas

$$\dot\varphi = \frac{\sqrt{\mu a(1-e^2)}}{r^2} \qquad (3)$$

where μ is the proportionality coefficient of the law of attraction.

The position of the orbit plane is defined by the longitude of the rising node Ω, which gives the direction of the unit vector **n** of the node line, and by the angle of inclination i of the orbit plane to the plane $O\xi\eta$ of the system of fixed axes $O\xi\eta\zeta$; the position of the perigee in the orbit plane is given by the angular distance ω of the perigee from the node, so that $\cos\omega = \mathbf{n}\cdot\mathbf{i}_1$.

The velocity vector of the perturbed motion, as follows from (1), (2), (3) is equal to

$$\mathbf{v} = \dot{\mathbf{r}} = \sqrt{\frac{\mu}{a}}\frac{1}{\sqrt{1-e^2}}\left[\mathbf{e}_r e\sin\varphi + \mathbf{e}_\varphi(1+e\cos\varphi)\right] \tag{4}$$

and the acceleration vector

$$\mathbf{w} = \dot{\mathbf{v}} = -\frac{\mu}{r^2}\mathbf{e}_r \tag{5}$$

Following the method of variation of parameters, for vectors **r** and **v** we will retain the same expressions (1) and (4) for the perturbed motion as for the unperturbed one; but the elliptic elements of the orbit a, e, Ω, i, ω will not be constants but unknown functions of time. On account of change of angles Ω, i, ω in the perturbed motion, the angular velocity ω of the set $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_3$ will be equal to

$$\mathbf{k} = \mathbf{k}\dot{\Omega} + \mathbf{n}\frac{di}{dt} + e_3(\dot{\varphi} + \dot{\boldsymbol{\varphi}}) \tag{6}$$

where **k** is the unit vector on the axis $O\zeta$.

Its projections on the axes of the set $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_3$, are obtained from the known formulas

$$\begin{aligned}
\omega_r &= \dot{\Omega}\sin i\sin u + \frac{di}{dt}\cos u\\
\omega_\varphi &= \dot{\Omega}\sin i\cos u - \frac{di}{dt}\sin u\\
\omega_3 &= \dot{\Omega}\cos i + \dot{\omega} + \dot{\boldsymbol{\varphi}} = \omega'_3 + \dot{\varphi}
\end{aligned} \tag{7}$$

where $u = \omega + \phi$. Let us note that ϕ in these equations of perturbed motion is different from the value obtained from (3); the latter will be denoted by ϕ^0; generally the small zero superscript will denote values for the unperturbed motion below.

From formulas for differentiation of unit vectors we have

$$\begin{aligned}
\dot{\mathbf{e}}_r &= \boldsymbol{\omega}\times\mathbf{e}_r = -\omega_\varphi\mathbf{e}_3 + (\omega_3' + \dot{\varphi})\mathbf{e}_\varphi\\
\dot{\mathbf{e}}_\varphi &= \boldsymbol{\omega}\times\mathbf{e}_\varphi = \omega_r\mathbf{e}_3 - (\omega_3' + \dot{\varphi})\mathbf{e}_r\\
\dot{\mathbf{e}}_3 &= \boldsymbol{\omega}\times\mathbf{e}_3 = -\omega_r\mathbf{e}_\varphi + \omega_\varphi\mathbf{e}_r
\end{aligned} \tag{8}$$

Setting as a condition the following equations

$$\dot{\mathbf{r}} = \mathbf{v} = \mathbf{v}^{\circ}, \qquad \dot{\mathbf{v}} = \mathbf{w}^{\circ} + \mathbf{F} \tag{9}$$

where F is an additional force acting at a point in a perturbed motion, after carrying out the differentiation and considering (8), we arrive at the equations

$$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{e}_r\left(\frac{\partial r}{\partial \varphi}\dot{\varphi} + \frac{\partial r}{\partial a}\dot{a} + \frac{\partial r}{\partial e}\dot{e}\right) + r\left[(\omega_3' + \dot{\varphi})\,\mathbf{e}_\varphi - \omega_\varphi \mathbf{e}_3\right] =$$

$$= \left(\mathbf{e}_r\frac{\partial r}{\partial \varphi} + \mathbf{e}_\varphi r\right)\dot{\varphi}^{\circ} = \sqrt{\frac{\mu}{a}}\frac{1}{\sqrt{1-e^2}}\left[\mathbf{e}_r e\sin\varphi + \mathbf{e}_\varphi(1+e\cos\varphi)\right] = v_r\mathbf{e}_r + v_\varphi\mathbf{e}_\varphi \tag{10}$$

$$\dot{\mathbf{v}} = \left(\frac{\partial v_r}{\partial a}\dot{a} + \frac{\partial v_r}{\partial e}\dot{e} + \frac{\partial v_r}{\partial \varphi}\dot{\varphi}\right)\mathbf{e}_r + \left(\frac{\partial v_\varphi}{\partial a}\dot{a} + \frac{\partial v_\varphi}{\partial e}\dot{e} + \frac{\partial v_\varphi}{\partial \varphi}\dot{\varphi}\right)\mathbf{e}_\varphi +$$

$$+ v_r\left[-\omega_\varphi\mathbf{e}_3 + (\omega_3' + \dot{\varphi})\,\mathbf{e}_\varphi\right] + v_\varphi\left[\omega_r\mathbf{e}_3 - (\omega_3' + \dot{\varphi})\,\mathbf{e}_r\right] = -\frac{\mu}{r^2}\mathbf{e}_r + \mathbf{F} \tag{11}$$

From (10) we obtain three equations

$$\omega_\varphi = 0,\ \omega_3' + \dot{\varphi} = \dot{\varphi}^{\circ},\ -\frac{\partial r}{\partial \varphi}\omega_3' + \frac{\partial r}{\partial a}\dot{a} + \frac{\partial r}{\partial e}\dot{e} = 0 \tag{12}$$

The last of these equations will become explicitly

$$\omega_3' e\sin\varphi - \frac{\dot{a}}{a}(1 + e\cos\varphi) + \frac{2e + e^2\cos\varphi + \cos\varphi}{1-e^2}\dot{e} = 0 \tag{13}$$

Making use of relation (12), the equations obtained from the vectorial equation (11) can be written in the following form

$$\begin{aligned} -\frac{\dot{a}}{2a}e\sin\varphi + \frac{\dot{e}}{1-e^2}\sin\varphi - \omega_3' e\cos\varphi &= \sqrt{\frac{a}{\mu}}\sqrt{1-e^2}\,F_r \\ -\frac{\dot{a}}{2a}(1+e\cos\varphi) + \frac{\dot{e}}{1-e^2}(\cos\varphi + e) + \omega_3' e\sin\varphi &= \sqrt{\frac{a}{\mu}}\sqrt{1-e^2}\,F_\varphi \end{aligned} \tag{14}$$

$$\omega_r = \sqrt{\frac{a}{\mu}}\frac{\sqrt{1-e^2}}{1+e\cos\varphi}F_3$$

From the first equation (12) and the last equation (14), recalling the values (7) of the quantities ω_r and ω_ϕ, we find the equations of perturbed motion for the elements Ω and i

$$\frac{di}{dt} = \sqrt{\frac{a}{\mu}}\frac{\sqrt{1-e^2}}{1+e\cos\varphi}F_3\cos u,\quad \dot{\Omega}\sin i = \sqrt{\frac{a}{\mu}}\frac{\sqrt{1-e^2}}{1+e\cos\varphi}F_3\sin u \tag{15}$$

From (13) and (14) we obtain

$$\begin{aligned} \dot{e} &= \sqrt{\frac{a}{\mu}}\sqrt{1-e^2}\left(F_r\sin\varphi + \frac{e + 2\cos\varphi + e\cos^2\varphi}{1+e\cos\varphi}F_\varphi\right) \\ \frac{\dot{a}}{2a} &= \sqrt{\frac{a}{\mu}}\frac{1}{\sqrt{1-e^2}}\left[F_r e\sin\varphi + (1+e\cos\varphi)F_\varphi\right] \\ \omega_3' &= \sqrt{\frac{a}{\mu}}\frac{\sqrt{1-e^2}}{e}\left(-F_r\cos\varphi + \frac{2+e\cos\varphi}{1+e\cos\varphi}F_\varphi\sin\varphi\right) = \dot{\Omega}\cos i + \dot{\omega} \end{aligned} \tag{16}$$

Equations (15), (16), together with the second equation (12), represent the required system of equations of perturbed motion.

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